Optimal Portfolio Selection for the Small Investor

Considering Risk and Transaction Costs[*]

Rainer Baule

University of Göttingen, Chair of Finance

Platz der Göttinger Sieben 3, D-37073 Göttingen, Germany

Phone: +49 551 39 - 8305, E-mail: rbaule@uni-goettingen.de

Abstract

A direct application of classical portfolio selection theory is problematic for the small investor, since transaction costs in the form of bank and broker fees exist. Particularly minimum fees force the investor to choose a rather small selection of assets. This leads to an optimization problem which juxtaposes the transaction costs against the risk costs arising with portfolios consisting of only a few assets. Despite the non-convex and thus complex optimization, an algorithmic solution turns out to be very fast and precise. An empirical study shows that for smaller investment volumes, transaction costs dominate risk costs, so that optimal portfolios contain only a very small number of assets.

Keywords: portfolio selection, transaction costs, non-convex optimization

JEL Classification: C 61, G 11

1 Introduction

Classical portfolio selection theory calls for the investor to diversify his portfolio and divide his wealth among a large number of securities. If the private investor with limited investment volume seeks to do this, he is forced to spread his purchasing power so thin that most of the assets enter his portfolio with only a few dollars invested. While optimal in a theoretical sense, in practice this procedure would be very expensive because of the transaction costs associated with each security bought. Hence the optimal portfolio selection in the presence of transaction costs might differ from the classical solution. Particularly for the small investor, transaction costs in the form of minimum fees play a decisive role, since they appear with each transaction, independent of its size. These minimum fees make the transaction cost function nonlinear, producing a comparably complex optimization problem.

However, the finance industry offers a broad range of products which promise the small investor to reap the benefits of diversification without the curse of high transaction costs due to small investment amounts. A very popular alternative to a direct investment in single stocks are mutual funds. But although direct transaction costs are kept to a minimum, there are other types of costs associated with investments in mutual funds, particularly premiums and periodic management fees. They are smaller for passively managed index tracking funds, but are still not negligible.

A second alternative are index certificates. These securities, which are issued by a growing number of banks, entitle the holder to participate directly at the performance of the underlying index, that is, to receive an amount of money equivalent to the value of the index at maturity. As issuers usually sell index certificates at a price equal to the current index value, they seem to be free of bank fees and hence the best alternative. However, many indices, such as the EUROSTOXX 50, are price indices, which are calculated without a correction for paid dividends. This involves implicit issuer fees of some 2–3 % annually, which makes the certificates comparably less attractive. A second type of hidden costs, even for return indices such as the German DAX, arises from the fact that unlike mutual funds, index certificates are unsecured

debt of the issuer and hence subject to its credit risk.[1] Although the majority of issuers has a high investment-grade credit rating, the negative value of credit risk is in the range of some dozens of base points.[2]

In the past years, a new financial innovation has evolved which combines the advantages of passively managed mutual funds and index certificates: the exchange-traded funds. These are funds with lean management and thus cost structures, which are listed at a special market segment of a stock exchange. In the case of index funds, they exhibit very competitive management fees. Furthermore, the funds are separated from the assets of the investment company, so they face no risk of default.

Nonetheless, even for exchange-traded funds, transaction costs are not zero, and the investor is furthermore restricted to the passive replication of the index. Thus, there is still a demand for a closer look at the portfolio selection problem of the small investor who intends to invest into single stocks rather than one of the mentioned alternatives. In particular, we assume that the investor has knowledge about the (joint) distribution of the single stock returns (which might be different from common beliefs of the market) and seeks to collocate his optimal portfolio. Even if he was able to purchase low-cost index products, his information might imply to follow a different strategy than replicating the index.

Based upon the seminal works of Markowitz (1952) and Tobin (1958), a number of contributions in the 1970s provide some insight into the portfolio selection problem when the number of securities in the portfolio is limited. Mao (1970) analyzes homogenous portfolios with different numbers of securities and finds that relatively few securities are needed to gain most of the benefits of diversification. Jacob (1974) formulates a portfolio selection problem with a cardinality-constraint, i.e. a restriction in the number of securities. Brennan (1975) explicitly accounts for transaction costs and derives the optimal number of securities in the portfolio for the Sharpe (1963) index model.

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[1] See Stoimenov/Wilkens (2005).

[2] See Baule/Entrop/Wilkens (2008).

The major part of the literature treating portfolio optimization in the presence of transaction costs focuses on linear or at least continuous cost functions. For the case of fixed transaction costs, Patel/Subrahmanyam (1982) and Blog et al. (1983) present algorithms for special forms of the covariance matrix of returns. In the past decade, some contributions address discrete constraints in the form of transaction costs or lots mainly under computational issues. Sankaran/Patil (1999) derive an algorithm for the cardinality-constraint portfolio selection with homogenous correlations. Bienstock (1996) applies branch-and-bound methods for this problem in the general case. Maringa/Kellerer (2003) propose simulated annealing. Several other heuristics including taboo search, threshold accepting, and genetic programming are applied to similar problems by Mansini/Speranza (1997), Bertsimas/Darnell/Soucy (1999), Gilli/Këllezi (2002), Chang (2001) and Jobst et al. (2001). Recently, Lobo/Fazel/Boyd (2007) address the problem of fixed transaction costs and propose an efficient heuristic algorithm for the maximization of the expected portfolio return subject to a number of various constraints.

However, little has been said about the particular optimization problem for the small investor who seeks to maximize his expected utility. In the light of transaction costs, the optimal portfolio will contain only a selection of all available assets compared to classical Markowitz theory. Due to this discrepancy, these portfolios harbour a measure of unsystematic risk which is not rewarded. In this paper we formulate the optimization problem in terms of costs arising from bearing unsystematic risk on the one hand and the transaction costs on the other hand. We demonstrate how this problem can be solved algorithmically. In the empirical part we analyze the trade-off between risk costs and transaction costs for an exemplary investment universe covering the EUROSTOXX 50 stocks, with a view to the relation between investment volume and transaction cost size.

2 Theoretical Model

2.1 Risk Costs versus Transaction Costs

We consider portfolio optimization in an extended Markowitz framework. Particularly, we reflect on an individual investor who maximizes his single-period expected utility. The corresponding preference function is expressed solely in terms of expected return and variance, which can be justified by either a quadratic utility function or normally distributed asset returns.[3] In the case of normal returns and exponential utility, the preference function can be stated as

$$u(\mu, \sigma^2) = \mu - \gamma \sigma^2, \tag{1}$$

where $\gamma \geq 0$ is the risk aversion parameter.[4]

There are n risky assets with random return X_i with expectation μ_i and standard deviation σ_i, $i = 1, \ldots, n$. Furthermore, there is a riskless asset with return r. The investor wishes to invest a fixed amount of money x in these assets—that is, choose a portfolio with (random) terminal wealth

$$P = x_0(1 + r) + \sum_{i=1}^{n} x_i(1 + X_i), \tag{2}$$

subject to the budget constraint

$$\sum_{i=0}^{n} x_i = x. \tag{3}$$

Furthermore, we introduce short-selling restrictions, which is reasonable for the private investor. Hence, the single investments must be non-negative, that is,

$$x_i \geq 0 \quad \text{for } i = 1, \ldots, n. \tag{4}$$

Associated with any investment $x_i > 0$ are transaction costs such as bank or broker fees. The simplest type of transaction costs are linear costs. However, it is very common in banking practice for minimum fees to apply with each transaction, an aspect that is particularly relevant for the small investor. Hence, a more realistic modelling of transaction costs should take this

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[3]See e.g. Haugen (2003), p. 201-204.

[4]This is consistent with an utility function $U(z) = -e^{-2\gamma z}$, see e.g. Sargent (1987), p. 154f.

nonlinearity into account. In this paper we allow for transaction costs with minimum fees of the following type:

$$c(x_i) = \max\{a, b\, x_i\} 1_{\{x_i > 0\}} \quad \text{with } a, b > 0. \tag{5}$$

Within this type of function, costs are constant at the minimum level up to the amount $x_i = a/b$. Above this amount, costs increase linearly. The indicator function $1_{\{x_i > 0\}}$ models the discrete jump from 0 to a when the investment volume increases from zero to greater zero, i.e. when a security enters into the portfolio.

Transaction costs lower the initial investment volume, or equivalently, lower the expected portfolio return. We assume that the riskless asset is free of transaction costs. This is reasonable since for instance German government bills can be purchased without costs even in small amounts. Thus, (2) becomes

$$P = x_0(1 + r) + \sum_{i=1}^{n} (x_i - c(x_i))(1 + X_i). \tag{6}$$

The investor seeks to maximize the preference function of the portfolio's expected return and variance, $u(E[P]/x - 1, Var[P]/x^2)$.

It is well known since Markowitz and Tobin that in the absence of transaction costs, the set of efficient portfolios consists of linear combinations of a tangential portfolio T and the riskless asset. That is, there is a parameter λ_C^{T},[5] the Sharpe ratio of the tangential portfolio, so that for a given σ the optimal expected return $\mu_C^T(\sigma) = r + \lambda_C^T \sigma$ can be achieved. Let σ_C^* be the optimal choice regarding the investor's preference function in absence of transaction costs, i.e.

$$(\mu_C^*, \sigma_C^*) = \arg\max\; u(\mu_C^T(\sigma), \sigma). \tag{7}$$

If transaction costs of the form (5) with a discrete component have to be taken into account, it will generally no longer be optimal to buy this portfolio. Rather, portfolios with a smaller number of assets and thus lower transaction costs might be more advantageous.

The crucial point of the transaction cost function is the discrete component: Transaction costs jump by a finite amount if an asset is taken into the portfolio even at infinitesimally small

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[5]The subscript C indicates classical Markowitz-Tobin portfolios.

volume. This component is responsible for the problem being no longer convex, so that classical methods cannot be applied.

However, if the set of assets with strictly positive weights in the portfolio is fixed, the transaction cost function is convex and thus classical portfolio selection on this subset of assets can be performed. Hence, a theoretical approach to solve the optimization problem would be to conduct a classical portfolio selection for each of the 2^n possible subsets.[6] Accordingly, the set of efficient portfolios is again the set of linear combinations of the riskless asset and a new tangential portfolio, which is calculated by the classical method but on a subset of all risky assets. The situation is demonstrated graphically in Figure 1.[7]

[Insert Figure 1 about here]

The new tangential portfolio is suboptimal in the sense of its Sharpe ratio λ^T being lower than λ_C^T. Furthermore, the remaining transaction costs lower expected portfolio returns. Together, the expected return of a portfolio on the new efficient line becomes

$$\mu = \mu^T(\sigma) = r + \lambda^T \sigma - \phi, \tag{8}$$

where

$$\phi = \sum_{i=1}^{n} \frac{c(x_i)}{x}(1 + r + \lambda^T \sigma) \tag{9}$$

are the relative transaction costs with respect to the total investment volume, compounded to the end of the period.

From the iso-preference curves $\mu - \gamma \sigma^2 = $ const., the first-order condition for the optimal portfolio is easily derived as

$$\frac{\partial \mu}{\partial \sigma} = 2\gamma\sigma \overset{!}{=} \lambda^T \tag{10}$$

$$\Rightarrow \sigma^* = \frac{\lambda^T}{2\gamma}. \tag{11}$$

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[6] See Lobo/Fazel/Boyd (2007), p. 353.

[7] See also Levy (1978), p. 645, for a similar illustration.

Hence, the existence of transaction costs reduces the optimal preference value by the amount

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\begin{aligned}
u(\mu_C^T(\sigma_C^*), \sigma_C^*) - u(\mu^T(\sigma^*), \sigma^*) &= \lambda_C^T \sigma_C^* - \gamma(\sigma_C^*)^2 - \lambda^T \sigma^* + \gamma(\sigma^*)^2 + \phi \\
&= \lambda_C^T \frac{\lambda_C^T}{2\gamma} - \gamma \left(\frac{\lambda_C^T}{2\gamma} \right)^2 - \lambda^T \frac{\lambda^T}{2\gamma} + \gamma \left(\frac{\lambda^T}{2\gamma} \right)^2 + \phi \\
&= \frac{(\lambda_C^T)^2}{4\gamma} - \frac{(\lambda^T)^2}{4\gamma} + \phi \\
&= (\lambda_C^T - \lambda^T) \frac{\lambda_C^T + \lambda^T}{4\gamma} + \phi \\
&= (\lambda_C^T - \lambda^T) \frac{\sigma_C^* + \sigma^*}{2} + \phi.
\end{aligned}
\tag{12}
$$

While the second component clearly comprises the transaction costs, the first component can be identified as a second type of costs: It describes the decrease in return for bearing unsystematic risk which is not rewarded. Hence, we interpret this type of opportunity costs as the risk costs associated with the portfolio.

These two types of costs obviously depend negatively on each other: Higher transaction costs correspond to a larger number of securities in the portfolio, thus a (possibly) higher Sharpe ratio of the tangential portfolio and lower risk costs. On the other hand, low transaction costs only allow for a small number of securities and hence a larger deviation of the tangential portfolio from the classically optimal one in terms of the Sharpe ratio.

2.2 Portfolio Optimization

As has become clear in the previous section, portfolio optimization in the presence of transaction costs breaks down to minimizing the sum of transaction costs and risk costs. Once a subset of securities is fixed, the corresponding tangential portfolio and thus the costs are easily calculated. The crucial task is to determine the subset of portfolios which minimizes these costs.

To find the optimal subset, a combinatorial search has to be performed. Each single security can either be in the portfolio or not, so there are 2^n possible combinations. If n is not very small, a combinatorial search is not realistic.[8] The problem is NP-hard, hence there is no

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[8]The numerical effort could be reduced by techniques like branch-and-bound, see Lobo/Fazel/Boyd (2007), p. 353.

algorithm which solves the problem in polynomial time.[9] However, there often are algorithms or heuristics which do not find the optimum but an approximate solution which can be very close to the optimum.[10] In this paper we adapt a heuristic proposed by Lobo/Fazel/Boyd (2007) for a similar portfolio selection problem. Their basic idea is to solve a sequence of convex problems related to the original non-convex problem. The solutions of this problems converge to a solution which is near the optimum of the original problem.

Transferred to our setting, the algorithm reduces to solving classical portfolio selection problems with adjusted expected returns. The adjustment in returns reflect the transaction costs as described above. Let μ_i^k denote the expected return for the i-th security within the k-th iteration step. Let furthermore $w^k = (w_i^k)_{i=1,\ldots,n}$, denote the vector of weights of the tangential portfolio for the classical portfolio selection problem with expected returns μ_i^k in the absence of transaction costs. The weights correspond to the respective investment volumes via $w_i^k = x_i^k/x$. The algorithm is then defined as follows:

- Define $\mu_i^0 = \mu_i - b$ and calculate w^0

- For each $k \geq 0$:

  - Define $\mu_i^{k+1} = \mu_i - \max\left\{b, \frac{a}{(w_i^k + \delta)x}\right\}$ and calculate w^{k+1}

  - If $||w^{k+1} - w^k|| < \epsilon$: STOP

Hereby, δ and ϵ are small, non-negative constants and $||\cdot||$ is a suited vector norm.

Within each step, the transaction costs are linearized to

$$c(x_i^{k+1}) = \max\left\{b, \frac{a}{(w_i^k + \delta)x}\right\} x_i^{k+1} = \max\left\{bx_i^{k+1}, \frac{ax_i^{k+1}}{(w_i^k + \delta)x}\right\}. \tag{13}$$

The rationale behind this approach is that if the procedure converges, $w_i^{k+1} = x_i^{k+1}/x$ will approximately equal w_i^k, so that if $w_i^{k+1} >> \delta$,

$$c(x_i^{k+1}) \approx \max\{a, bx_i^{k+1}\} \tag{14}$$

[9]See Bienstock (1996).

[10]Besides the deterministic algorithm which is described in the following, probabilistic algorithms like genetic programming promise some success, see Chang et al. (2000).

as desired. The performance of the algorithm is fairly good. Usually it converges after 10–20 iterations. The error with respect to the true solution in terms of the preference function is lower than 0.01%.[11]

3 Empirical Investigation

3.1 Data

In this section, the theoretical results are applied to an exemplary optimization problem of an investor whose universe consists of the 50 blue chips of the EUROSTOXX 50. Of main interest is the question of how the number of stocks in the optimal portfolio depends on the transaction costs or the investment volume, respectively. Furthermore, we analyze the proportion of transaction costs and risk costs.

To obtain realistic input data, the parameters are estimated from historical market prices. Therefore, we use a two-year history (from 02/09/2002 to 03/09/2004) of closing quotes for the 50 stocks at the Frankfurt stock exchange. We estimate the variance-covariance-matrix directly using a standard approach. The vector of expected returns is estimated implicitly by applying the capital asset pricing model.[12] Therefore, we use the two-year history to estimate the beta factors β_i for each stock with respect to the index and calculate the implied expected return via

$$\mu_i = r + \beta_i(\mu_M - r). \tag{15}$$

The market premium $\mu_M - r$ is assumed to be equal to 6.5%, which represents a consensus of leading academic financial economists (see Welch (2000)). The risk-free rate is identified with the value $r = 2.2\%$ as of November 2004.

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[11] As will become clear in the empirical part, in many cases buying a very small number of assets is optimal. With a given bound on the number of assets, a complete combinatorial search to find the exact solution becomes achievable again. Calculating the exact solution in every case where the optimal number of assets is bounded by 4 (out of 50) yields the quoted figure.

[12] Jorion (1991) reports evidence that this kind of estimation leads to comparably good results.

To get an idea of the size of transaction costs, Table 1 gives a brief overview of fees of five major banks and discount brokers in Germany. These fees apply for internet transactions, whereas fees for personal transactions are usually higher. However, it is reasonable to assume in our analysis that the investor seeks to minimize costs and thus chooses the cheapest order channel.

[Insert Table 1 about here]

It is evident that there are significant differences between this representative selection of banks and brokers. Furthermore, besides the minimum fee and a linear component, several other variations exist for these and other offerers, particularly maximum fees, degressive shapes, discrete jumps etc. However, as mentioned, for our purpose the discrete minimum fee is of main interest, so that the assumed function type (5) is a good model for real transaction costs. For our empirical study we apply a minimum fee of 10 Euro and a linear component of $0.25\,\%$, which are representative values according to Table 1.

3.2 Results

Figure 2 shows the risk-return-profile of the 50 stocks and the corresponding efficient frontier. It becomes evident that the straight line of efficient portfolios is very close to the curved boundary without the riskless asset in quite a large range. This observation refers back to the assumption made to calculate the expected returns: Since we assume the capital asset pricing model holds, all single stocks lie perfectly along the security market line. As, furthermore, the stocks are fairly positively correlated, a major part of stock variances is explained by the market. Hence, their positions in the μ-σ-diagram are densest along a straight line parallel to the efficient line (which is identical to the capital market line in this case), which explains the shape of the curved boundary. The tangential portfolio has an expected return of $8.4\,\%$ and a standard deviation of $26.1\,\%$.

[Insert Figure 2 about here]

Based upon this data set, we calculate optimal portfolios with respect to transaction costs and total investment volume. Optimal tangential portfolios and thus optimal subsets of assets

are independent of individual preferences. The latter however influence the absolute size of transaction and risk costs because they determine the amount invested into the riskless asset and hence the position of the optimal portfolio on the efficient line. The following analysis is based upon an average investor whose preference function implies that the tangential portfolio is optimal in the absence of transaction costs.[13] Hence, we calculate portfolios with maximum return, given the standard deviation $\sigma_C^* = 26.1\%$. The results are summarized in Table 2.

[Insert Table 2 about here]

It becomes evident that for very small total investment volumes up to 2,000 Euro, the optimal number of assets is as small as one. If only one single stock has to be chosen, it is optimal to invest in the stock with the largest ratio $(\mu_i - r)/\sigma_i$,[14] since this stock is the best substitute for the market portfolio. In doing so, the risk costs remain below one percent. With three stocks, the risk costs decrease to 0.22%, which is optimal up to an investment volume of 20,000 Euro. The optimal average transaction size increases relatively slowly with the investment amount. For a wide range of investment volumes, values of 5,000 to 10,000 Euro are optimal.[15]

With the exception of the smallest volume, relative transaction costs are fairly independent of the investment volume at values between 0.2% and 0.3%. (Note that they can fall below the assumed linear transaction cost component of 0.25%, because the optimal portfolio may involve investing into the riskless asset which is considered to be free of transaction costs.) Naturally, the risk costs decrease with the number of assets, so for larger investment volumes above 20,000 Euro, the transaction costs become the major part of the total costs.

These results are based on one representative example. The findings concerning the optimality of only few securities can be drawn back to the shape of the covariance matrix, which exhibits moderate to rather high levels of correlations among the single stocks. This leads to a limited

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[13] This is equivalent with an absolute risk aversion of $2\gamma = (\mu_M - r)/\sigma_M^2 = 0.954$.

[14] Provided the absence of estimation errors, this is the stock with the highest Sharpe ratio.

[15] The observation that only very few securities can be optimal is in line with results of Brennan (1975) within his index model.

potential for diversification; in other words, a major part of the total diversification is already achieved by a small number of stocks. This statement relaxes if the covariance matrix would be more diagonal dominant. In this case, due to lower correlations, a higher level of diversification would be achievable and thus a larger number of securities would be optimal also in the presence of transaction costs. However, the covariance matrix estimated for the EUROSTOXX 50 can be seen as typical, so the general results presented here will also hold for other markets.

4 Conclusion

The existence of transaction costs forces the small investor to concentrate his money on a rather small selection of securities. We analyzed the resulting optimization problem in more detail by reformulating it as the minimization of the sum of two types of costs, the transaction costs and the risk costs. Risk costs arise with bearing unsystematic risk, which is unavoidable if the held portfolio differs from the optimal portfolio in classical Markowitz theory. Because of transaction costs, however, the small investor seeks to reduce the number of assets compared to this classically optimal portfolio.

The optimization problem is NP-hard. We propose an efficient heuristic which proves to be very powerful in terms of computational speed as well as accuracy of the found solution. The empirical analysis showed that the existence of transaction costs can lead to a very small optimum for the number of stocks in the portfolio.

Our analysis also provides guidelines for choosing between direct investments into the stocks and alternatives discussed in the introduction: The direct investment is advantageous if the sum of associated risk and transaction costs is smaller than the annual costs arising with the alternatives. Furthermore, an alternative is only reasonable if it is based on the same information of the investor. For instance, if the investor has private information which are suited to outperform the market, it is not reasonable to undertake a passive index investment. Based on the assumption that the investor's information imply to replicate the index, the cost structures of (classical) mutual index funds, index certificates, and exchange-traded index funds

can directly be compared with the sum of risk and transaction costs of a direct investment. As the risk costs can be bounded by 0.8% even with only one single stock in our example, and transaction costs decrease sharply with the investment volume, this figure represents the upper limit for the annual fee of a fund—if it is to have any chance of being advantageous. It has to be mentioned that not few (even passively managed) mutual funds exceed this threshold. For index certificates the negative value of credit risk has to be considered, which is below this level for most of the issuers. Also exchange-traded index funds exhibit low cost structures (with annual fees down to $0.1\% - 0.2\%$), which makes them a worthwhile vehicle for small investors. Usually there are also transaction costs associated with these investments, which have to added to these fees. Finally the choice between a direct investment and one of the alternatives depends on the investment volume. Based on our analysis, for given percentage costs of an alternative, a lower level for the investment volume can be derived, above which a direct investment is advantageous.

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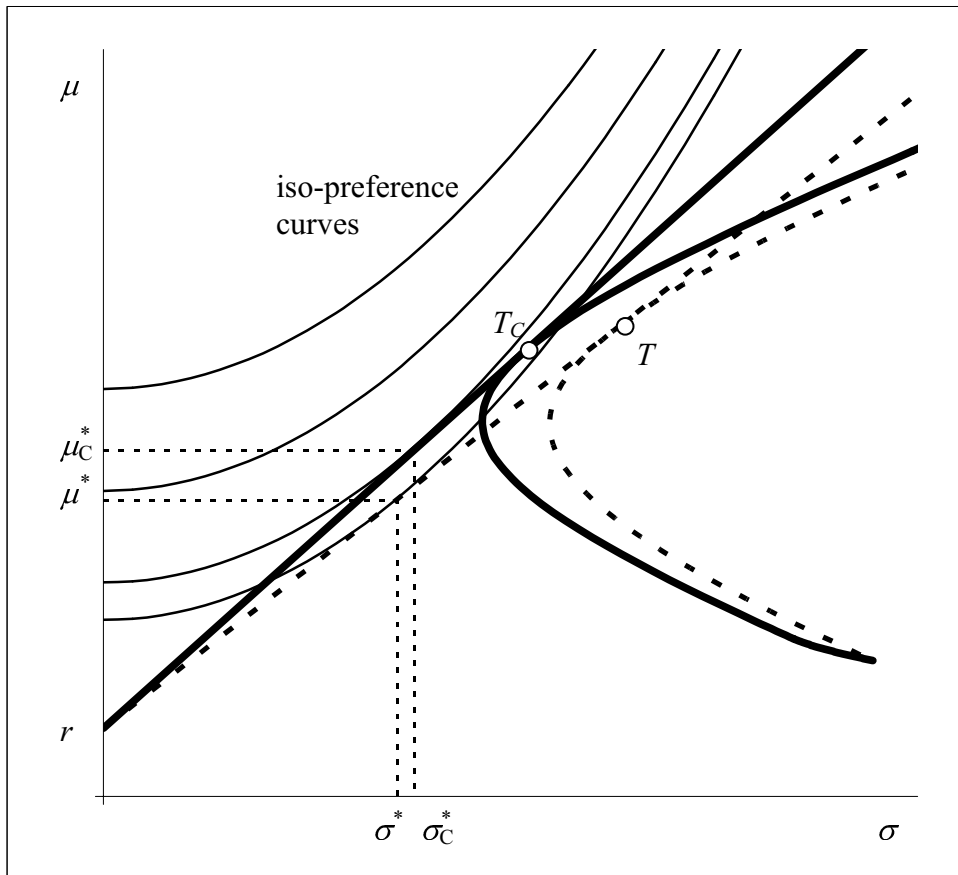
Figure 1. *Portfolio optimization with and without transaction costs: Without transaction costs, the solid efficient frontier and the tangential portfolio T_0 can be achieved. In the presence of transaction costs, only a selection of assets enters into the portfolio, which leads to the dotted efficient frontier and the tangential portfolio T. For a given preference function, the parameters of the optimal portfolio change from μ_C^* to μ^* and from σ_C^* to σ^*, respectively.*

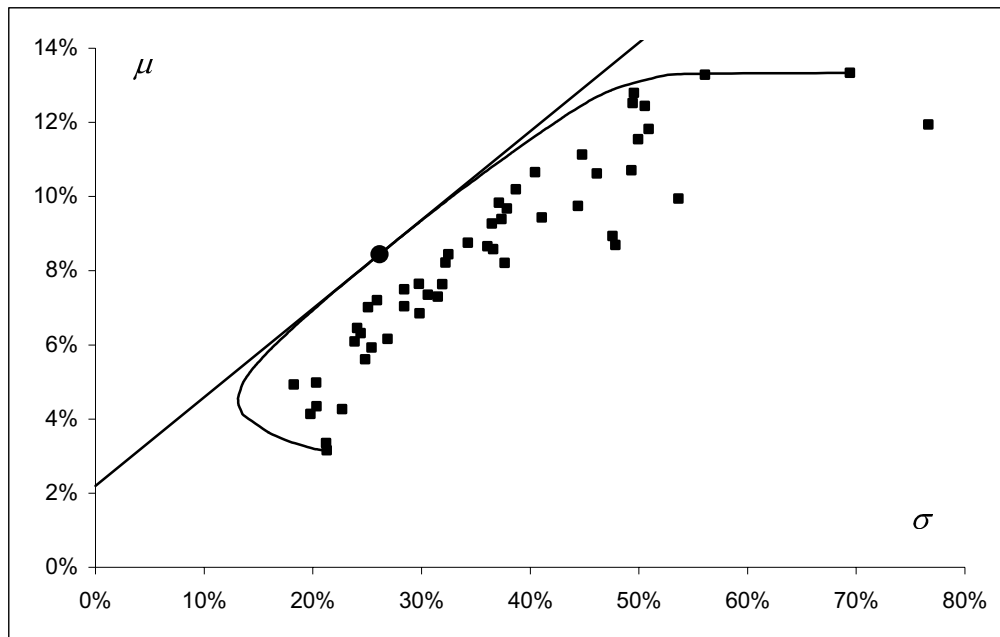**Figure 2.** μ-σ-diagram for the EUROSTOXX 50: The small squares represent the single stocks. Due to the CAPM assumption the curved efficient frontier is very close to the efficient line. The parameters of the tangential portfolio (circle) are $\mu = 8.4\ \%$ and $\sigma = 26.1\ \%$.

	minimum fee a	linear component b
Citibank AG	9.99 Euro	0.00 %
comdirect bank AG	9.90 Euro	0.25 %
Commerzbank AG	24.90 Euro	1.00 %
Cortal Consors S.A.	9.95 Euro	0.25 %
Deutsche Bank AG	20.00 Euro	1.00 %

Table 1. *Transaction cost structures for five banks and brokers in Germany as of February 2005.*

x	n^*	$\overline{x}_i$	ϕ	ψ	$\frac{\phi}{\phi+\psi}$
1,000	1	1,000	0.53 %	0.79 %	40 %
2,000	1	2,000	0.26 %	0.79 %	25 %
5,000	2	2,500	0.32 %	0.39 %	45 %
10,000	3	3,300	0.24 %	0.22 %	52 %
20,000	3	6,700	0.20 %	0.22 %	47 %
50,000	6	8,300	0.19 %	0.11 %	63 %
100,000	14	7,100	0.21 %	0.04 %	84 %
200,000	19	10,500	0.21 %	0.03 %	88 %
500,000	29	17,200	0.21 %	0.02 %	92 %

Table 2. *Properties of optimal portfolios: investment volume x, optimal number of assets n^*, average transaction size $\overline{x}_i$, relative transaction costs ϕ, relative risk costs ψ, transaction costs relative to total costs $\phi/(\phi+\psi)$.*